# ON POLYNOMIALS IN SELF-ADJOINT OPERATORS IN SPACES WITH AN INDEFINITE METRIC[1]

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1. **Introduction.** Let $H$ be a Hilbert space[2] with the usual inner product $[\cdot, \cdot]$ and norm[3] and with an indefinite inner product $(\cdot, \cdot)$ which, for some orthogonal decomposition $H = H_1 \oplus H_2$ in $H$, is defined by

$$(x, y) = [x_1, y_1] - [x_2, y_2] \quad \text{for all } x, y \in H,$$

where

$$x = x_1 + x_2, \qquad y = y_1 + y_2,$$
$$x_1, y_1 \in H_1, \qquad x_2, y_2 \in H_2,$$

and $\dim H_1 = \kappa$, a positive integer. Such a space $H$ will be called a space $\Pi_\kappa$ with an indefinite metric. Another, axiomatic definition of the space $\Pi_\kappa$ was given by I. S. Iohvidov and M. G. Krein in [1]; we follow here their terminology, unless otherwise stated, and use the results of their paper.

A linear operator $A$ in $\Pi_\kappa$ is called symmetric if it maps a dense domain $D(A)$[4] in $\Pi_\kappa$ into $\Pi_\kappa$ and has the property,

$$(Ax, y) = (x, Ay) \quad \text{for all } x, y \in D(A).$$

A linear operator $A^*$ defined in $\Pi_\kappa$ is called the adjoint operator of a linear operator $A$ with a dense domain $D(A)$ in $\Pi_\kappa$ if $A^*$ is the maximum operator such that

$$(Ax, y) = (x, A^*y) \quad \text{for all } x \in D(A) \text{ and all } y \in D(A^*).$$

A symmetric operator is said to be maximal if it has no proper symmetric extension.

A symmetric operator is said to be self-adjoint if $A = A^*$.

It is well known in the theory of operators in Hilbert spaces that any two complex conjugate polynomials in a self-adjoint operator are adjoint to each other. We find that the same property holds for polynomials in a self-adjoint operator in the space $\Pi_\kappa$ with an indefinite metric. Moreover, if there exists a pair of complex conjugate

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(1) This paper is supported by Canadian National Research Council. Part of the results are from the author's Ph.D. thesis.

(2) $H$ is not necessarily separable.

(3) The topology in this paper is the norm topology.

(4) We shall always denote the domain of an operator $A$ by $D(A)$.

polynomials in a symmetric operator one of which is adjoint to the other, then this operator is self-adjoint. We shall prove these assertions in this paper.

2. **Closed isometric operators.** We shall prove here a theorem on isometric operators for later use. Isometric operators in $\Pi_\kappa$ are, in general, not continuous. However, a closed isometric operator in $\Pi_\kappa$, as we shall show, is continuous.

DEFINITION 2.1. A linear operator $V$ is said to be *isometric* if

$$(Vx, Vx) = (x, x) \quad \text{for all } x \in D(V).$$

DEFINITION 2.2. Let $\Pi_\kappa = P \oplus N$, where $P$ is a positive $\kappa$-dimensional subspace and $N$ is the orthogonal complement of $P$. An operator $J$ is called a *metric operator* if it is defined by the relation,

$$J(x) = x_P - x_N \quad \text{for all } x \in \Pi_\kappa$$

where $x = x_P + x_N$, $x_P \in P$ and $x_N \in N$. The new scalar product $[x, y]_J = (x, Jy)$ is called a *J-metric* and the new norm $|x|_J = ([x, x]_J)^{1/2}$ is called a *J-norm*. By Theorem 1.2 in [1, §2] all the $J$-norms are topologically equivalent.

NOTATION. For any two linear manifolds $L$, $M$ the notation $L \oplus M$ represents that $(x, y) = 0$ for all $x \in L$ and all $y \in M$. The notation $L \oplus_J M$ represents that $L \oplus M$ and $[x, y]_J = 0$ for all $x \in L$ and all $y \in M$ for some metric operator $J$.

THEOREM 2.3. *If $V$ is a closed isometric operator, then $V$ is a continuous operator with a closed domain $D(V)$ and a closed range $R(V)$.*

**Proof.** Let $D_+(V)$ be a positive subspace of $D(V)$ with the greatest possible dimension and $R_+(V) = VD_+(V)$. Since $V$ is an isometric operator, the subspace $R_+(V)$ is a maximal positive subspace of $R(V)$, having the same dimension as $D_+(V)$. We can have the resolutions

$$D(V) = D_+(V) \oplus_J D_-^0(V) \quad \text{and} \quad R(V) = R_+(V) \oplus_{J'} R_-^0(V)$$

where $D_-^0(V)$ and $R_-^0(V)$ are nonpositive orthogonal complements of $D_+(V)$ and $R_+(V)$ in $D(V)$ and $R(V)$ respectively. If the scalar product degenerates on $D_-^0(V)$, then by Theorem 1.7 in [1, §3] we have

$$D_-^0(V) = D_0(V) \oplus_J D_-(V),$$

where $D_0(V)$ is the isotropic subspace of the linear manifold $D_-^0(V)$ and $D_-(V)$ is a negative linear manifold. Clearly $R_0(V) = VD_0(V)$ is the isotropic subspace of the linear manifold $R_-^0(V)$ and $R_-(V) = VD_-(V)$ is a negative linear manifold. Obviously we have

$$R_-^0(V) = VD_0(V) \oplus VD_-(V).$$

Thus we have the resolutions,

$$D(V) = D_+(V) \oplus_J D_0(V) \oplus_J D_-(V) \tag{2.1}$$

and

$$R(V) = R_+(V) \oplus_{J'} R_0(V) \oplus R_-(V). \tag{2.2}$$

If cl $(R_-(V))$, the closure of $R_-(V)$, is a negative subspace then the theorem is a direct consequence of Theorem 4.3 in [1, §15]. Thus it remains to prove that cl $(R_-(V))$ is a negative subspace.

Assuming that the nonpositive subspace cl $(R_-(V))$ is a degenerate subspace, by Theorem 1.7 in [1, §3] we have the decomposition

$$\text{cl}\,(R_-(V)) = N \oplus_{J'} R'_- \tag{2.3}$$

where $N$ is the isotropic subspace of cl $(R_-(V))$ and $R'_-$ is a negative subspace. Similarly we have the decomposition

$$\text{cl}\,(D_-(V)) = M \oplus_J D'_-, \tag{2.4}$$

where $M$ is the isotropic subspace of cl $(D_-(V))$ and $D'_-$ is a negative subspace.

Now let $z_0 \in N$. Then there exists a sequence $\{x_n\}_0^\infty$ in $D_-(V)$ such that $\{y_n = Vx_n\}_0^\infty$ is a Cauchy sequence in $R_-(V)$, having $z_0$ as its limit. From (2.3) and (2.4) we have

$$x_n = x_{0n} + x'_n, \qquad y_n = y_{0n} + y'_n,$$

where $x_{0n} \in M$, $x'_n \in D'_-$, $y_{0n} \in N$ and $y'_n \in R'$ for $n = 0, 1, 2, \ldots$. Clearly we have

$$(x'_n, x'_n) = (x_n, x_n) = (y_n, y_n) = (y'_n, y'_n) \tag{2.5}$$

for $n = 0, 1, 2, \ldots$. Since the scalar product $(.\,, .)$ is continuous in both arguments we have

$$\lim_{n\to\infty} (y_n, y_n) = (z_0, z_0) = 0.$$

It follows from (2.5) that

$$\lim_{n\to\infty} [x'_n, x'_n]_J = -\lim_{n\to\infty} (x'_n, x'_n) = -\lim_{n\to\infty} (y'_n, y'_n)$$

$$= \lim_{n\to\infty} [y'_n, y'_n]_{J'} = -\lim_{n\to\infty} (y_n, y_n) = 0.$$

In other words, each of the sequences $\{x'_n\}_0^\infty$ and $\{y'_n\}_0^\infty$ converges to the zero vector $\theta$. Hence the sequence $\{y_{0n}\}_0^\infty$ converges to $z_0$.

If the sequence $\{x_{0n}\}_0^\infty$ has a Cauchy subsequence with a limit $x \in M$, then $z_0 = Vx$ and $x \in D_-(V)$ since $V$ is a closed operator. It follows that $z_0 \in N \cap R_-(V)$, that is $z_0 = \theta$.

If the sequence $\{x_{0n}\}_0^\infty$ had no Cauchy subsequence, then it would have an unbounded subsequence $\{x_{0k}\}_0^\infty$ such that $|x_{0k}|_J = d_k > k+1$ for $k = 0, 1, 2, \ldots$, since $M$ is a finite dimensional subspace by Lemma 1.2 in [1, §1].

We define a sequence

$$w_k = x_k/d_k = x_{0k}/d_k + x'_k/d_k \quad \text{for } k = 0, 1, 2, \ldots.$$

The sequence $\{w_k\}_0^\infty$ is clearly in $D_-(V)$ and the sequence $\{V(w_k) = y_k/d_k\}$ is clearly a Cauchy sequence in $R_-(V)$ with the limit $\theta$. The sequence $\{x'_k/d_k\}_0^\infty$ converges to $\theta$ and the sequence $\{x_{0k}/d_k\}_0^\infty$ is bounded in $M$ with $|x_{0k}/d_k|_J = 1$ for $k = 0, 1, 2, \ldots$.

Let $\{x_{0m}/d_m\}_0^\infty$ be a Cauchy subsequence of $\{x_{0k}/d_k\}_0^\infty$, with limit $w_0 \in M$. It follows that the corresponding subsequence $\{w_m\}_0^\infty$ is also a Cauchy sequence with limit $w_0$. Since $V$ is a closed operator, we have $V(w_0)=\theta$ and $w_0 \in D_-(V)$, that is $w_0=\theta$. But $|w_0|_J=1$ since $w_0$ is the limit of the sequence $\{x_{0m}/d_m\}_0^\infty$. This contradiction implies that the sequence $\{x_{0n}\}_0^\infty$ is bounded and hence $z_0=\theta$. In other words $N=\{\theta\}$. Now it is easy to show that $D(V)$ and $R(V)$ are closed. The theorem is proven.

3. **Polynomials in self-adjoint operators in the space** $\Pi_\kappa$. Having proven Theorem 2.3 we are now able to investigate some properties of a symmetric operator by using Cayley-von Neumann transformation. Since every symmetric operator has a closed symmetric extension (see §6 in [1]), we center our attention on closed symmetric operators.

Let $A$ be a closed symmetric operator with a dense domain $D(A)$. There exists a nonreal number $\zeta$ which is not a proper value of $A$ since a symmetric operator in $\Pi_\kappa$ can have at most $2\kappa$ nonreal proper values (see 1 of §8 in [1]). We define an operator $V$ by the following formulae:

$$y = (Ax-\zeta x), \qquad Vy = (Ax-\bar{\zeta}x) \quad \text{for } x \in D(A),$$

where $\bar{\zeta}$ is the complex conjugate of $\zeta$ or symbolically,

$$V = (A-\bar{\zeta}I)(A-\zeta I)^{-1} \quad \text{and} \quad D(V) = (A-\zeta I)D(A).$$

The operator $V$ is clearly a closed isometric operator. It follows from Theorem 2.3 that $V$ is a continuous operator with a closed domain $D(V)=(A-\zeta I)D(A)$. Now it is easy to see that the operator $(A-\zeta I)^{-1}$ is continuous. Thus we have proven the following theorem.

THEOREM 3.1. *Let $A$ be a closed symmetric operator with a dense domain $D(A)$. If $\zeta$ is a nonreal number which is not a proper value of $A$, then the operator $(A-\zeta I)^{-1}$ is continuous with a closed domain $(A-\zeta I)D(A)$.*

Before we prove our main theorem, we need to establish a few lemmas for later use.

LEMMA 3.2. *Let $A$ be a linear operator in a linear space $\Pi$ and let $\zeta$ be a complex number. If $(A-\zeta I)D(A^m)\supset D(A^m)$ for some positive integer $m$, then $(A-\zeta I)D(A^n) = D(A^{n-1})$ for all natural numbers $n>m$.*

**Proof.** We shall prove this lemma by induction. Let $n=m+1$. It is obvious that $(A-\zeta I)D(A^{m+1})\subset D(A^m)$. We need to prove only the reverse inclusion. For any $x \in D(A^m)$ by assumption there exists $y \in D(A^m)$ such that $(A-\zeta I)y=x$. It follows that $Ay \in D(A^m)$, that is $y \in D(A^{m+1})$. Hence $(A-\zeta I)D(A^{m+1})\supset D(A^m)$ and we have proved our lemma for $n=m+1$. Using the same kind of arguments we can prove the lemma for the case $n=k+1$ by assuming it is true for $n=k$. The lemma is proven.

LEMMA 3.3. *Let $A$ be a self-adjoint operator in $\Pi_\kappa$ and let $P(\lambda)=\prod_{i=1}^{n}(\lambda-\zeta_i)$ be a polynomial with nonreal roots. If no root of $P(\lambda)$ is a proper value of $A$, then*(5) *$P(A)D(A^m)=D(A^{m-n})$ for $m>n$, where $m$, $n$ are natural numbers.*

**Proof.** By Theorem 2.9 in [1, §9] we have $(A-\zeta_i I)D(A)=\Pi_\kappa$ for $i=1, 2, \ldots, n$. Thus this lemma follows Lemma 3.2 immediately.

LEMMA 3.4. *Let $A$ be a maximal symmetric operator in $\Pi_\kappa$. Then $D(A^n)$ is dense in $\Pi_\kappa$ for any natural number $n$.*

**Proof.** We shall prove this lemma by induction. For $n=1$ the lemma is true by the definition of a maximal symmetric operator.

Now we assume this lemma is true for $n=m$. By Theorem 2.9 in [1, §9] we have a pair of complex numbers $(\zeta, \bar{\zeta})$ such that

$$(A-\zeta)D(A) = \Pi_\kappa \quad \text{and} \quad (A-\bar{\zeta})D(A) = M, \tag{3.1}$$

where $M$ is a nondegenerate subspace, containing a $\kappa$-dimensional positive subspace. Thus by Theorem 1.5 in [1, §3] we have the resolution

$$\Pi_\kappa = M \oplus_J N, \tag{3.2}$$

where $N$ is the orthogonal complement of $M$. By Lemma 3.2 we have

$$(A-\zeta)D(A^{m+1}) = D(A^m) \tag{3.3}$$

from relation (3.1).

Now for any $x \in \Pi_\kappa$ we have $y \in D(A)$ such that $x=(A-\zeta)y=(A^*-\zeta)y$ by relation (3.1). From relation (3.2) we have $y=y_M+y_N$, when $y_M \in M$ and $y_N \in N$. It follows from (3.1) there exists $z \in D(A)$ such that $y_M=(A-\bar{\zeta})z$. Since $(A^*-\zeta)y_N=\theta$, we have

$$x = (A^*-\zeta)(A-\bar{\zeta})z \tag{3.4}$$

for some $z \in D(A)$. If $x \in \Pi_\kappa$ and $(x, D(A^{m+1}))=0$ then

$$\begin{aligned} 0 &= (x, D(A^{m+1})) = ((A^*-\zeta)(A-\bar{\zeta})z, D(A^{m+1})) \\ &= ((A-\bar{\zeta})z, (A-\bar{\zeta})D(A^{m+1})) \\ &= ((A-\zeta)z, (A-\zeta)D(A^{m+1})). \end{aligned}$$

It follows from (3.3) that $0=((A-\zeta)z, D(A^m))$. Hence we have $(A-\zeta)z=\theta$ by assumption. Since $\zeta$ is not a proper value of $A$ we must have $z=\theta$. Thus from (3.4) we conclude that $x=\theta$. It thus follows that $D(A^{m+1})$ is dense in $\Pi_\kappa$. The lemma is proved.

LEMMA 3.5. *Let $P(\lambda)$ be a polynomial of degree $n$ and let $F$ be a finite set of $m$ complex numbers. Then we can always find a nonreal number $\zeta_0$ such that all the roots of the polynomial $P(\lambda)-\zeta_0$ are nonreal and these roots are not in the set $F$.*

(5) We agree that for any operator $A$, $A^0=I$ where $I$ is the identity operator.

**Proof.** It is easy to see that if $\zeta$ and $\zeta'$ are different numbers, then the polynomials $P(\lambda)-\zeta$ and $P(\lambda)-\zeta'$ have no common factors. Hence for only a finite number of complex numbers $\zeta_i$, $i=1, 2, \ldots, m'$ $(m' \leqq m)$ does the corresponding polynomial $P(\lambda)-\zeta_i$ have roots in the set $F$. It thus follows that for any complex number $\zeta$ such that $\operatorname{Re} \zeta > \operatorname{Re} \zeta_i$, $i=1, 2, \ldots, m'$ the polynomial $P(\lambda)-\zeta$ has no roots in $F$.

Let $P(\lambda)=P^{(1)}(\lambda)+iP^{(2)}(\lambda)$, where $P^{(1)}(\lambda)$ and $P^{(2)}(\lambda)$ are real polynomials of degree at most $n$. Let $\zeta=c+id$, where $c \neq 0$ and $d$ are real numbers such that $c > \operatorname{Re}(\zeta_i)$, $i=1, 2, \ldots, m'$. If $\lambda_0$ is a real root of the polynomial $P(\lambda)-\zeta$, then we have

$$P^{(1)}(\lambda_0)-c = 0 \tag{3.5}$$

and

$$P^{(2)}(\lambda_0)-d = 0. \tag{3.6}$$

It is clear that for a fixed real number $c$, there exist at most $n$ $\lambda_0$'s satisfying the relation (3.5). It thus follows that we can find a real number $d_0 \neq 0$ such that the polynomial $P(\lambda)-(c+id_0)$ has no real roots. Hence the number $\zeta_0=c+id_0$ is the desired nonreal number. The lemma is proved.

LEMMA 3.6. *If $A$ is a closed linear operator in $\Pi_\kappa$ then the adjoint of the adjoint of $A$ is $A$.*

**Proof.** Let $J$ be a metric operator. Clearly $JA$ is also a closed linear operator since $J$ is a bicontinuous linear operator by Theorem 1.2 in [1, §2]. Let us denote the adjoint of $JA$ with respect to the $J$-metric by $(JA)^J$. Since the space $\Pi_\kappa$ together with a $J$-metric is a Hilbert space, we have $(JA)^{JJ}=JA$. It is obvious that for any linear operator $B$ with a dense domain $(JB)^J=JB^*$. It thus follows that $JA=(JA^*)^J$ $=JA^{**}$. Since $J$ is bijective, we have $A=A^{**}$. The lemma is proved.

THEOREM 3.7. *Let $A$ be a symmetric operator in $\Pi_\kappa$ and let $P(\lambda)$ and $\bar{P}(\lambda)$ be complex conjugate polynomials of degree $n$. Then the operator $\bar{P}(A)$ is adjoint to $P(A)$ if and only if $A$ is a self-adjoint operator.*

**Proof.** (1). Let $A$ be a self-adjoint operator. Since $A$ can have only a finite number of nonreal proper values, it follows that by Lemma 3.5 we can find a nonreal number $\zeta$ such that the polynomial $P(\lambda)-\zeta$ has no root which is a proper value of $A$ or its complex conjugate. Hence $\bar{P}(\lambda)-\bar{\zeta}$ also has no root which is a proper value of $A$. It follows that $\zeta$ and $\bar{\zeta}$ are not proper values of $P(A)$ and $\bar{P}(A)$ respectively.

It is clear that $D(P(A))=D(A^n)=\bar{D}(P(A))$. Since $D(A^n)$ is dense in $\Pi_\kappa$ by Lemma 3.4, the adjoint operator $P(A)^*$ of $P(A)$ exists. Obviously we have $P(A)^* \supset \bar{P}(A)$. Therefore it is sufficient to prove $D(A^n) \supset D(P(A)^*)$ in order to prove $\bar{P}(A)=P(A)^*$.

By Lemma 3.3 we have

$$(P(A)-\zeta I)D(A^n) = \Pi_\kappa = (\bar{P}(A)-\bar{\zeta} I)D(A^n). \tag{3.7}$$

For any $x \in D(P(A)^*)$ there exists $z \in D(A^n)$ such that

$$(P(A)^*-\bar{\zeta} I)x = (\bar{P}(A)-\bar{\zeta} I)z$$

by relation (3.7). In other words, we have $(P(A)^*-\bar{\zeta} I)(x-z)=\theta$. It thus follows that for all $y \in D(A^n)$ we have

$$0 = ((P(A)^*-\bar{\zeta} I)(x-z), y) = ((x-z), (P(A)-\zeta I)y).$$

Since $(P(A)-\zeta I)D(A^n)=\Pi_\kappa$, we have $x-z=\theta$, that is $x=z \in D(A^n)$. Similarly we can prove $\bar{P}(A)^*=P(A)$. The first part of the theorem is proved.

(2) Now let $P(A)$ and $\bar{P}(A)$ be adjoint to each other. We choose $\zeta$ such that the polynomial $P(\lambda)-\zeta=\prod_{i=1}^n (\lambda-\zeta_i)$ has no root which is a proper value or its complex conjugate of the operator $\bar{A}$, the closed extension of $A$. It thus follows $\zeta$ and $\bar{\zeta}$ are not proper values of $P(A)$ and $\bar{P}(A)$ respectively.

We shall show that $(P(A)-\zeta I)D(A^n)$ is dense in $\Pi_\kappa$. Let $x \in \Pi_\kappa$ be such that $(x, (P(A)-\zeta I)y)=0$ for all $y \in D(A^n)$. It follows that for all $y \in D(A^n)$ we have

$$0 = ((P(A)^*-\bar{\zeta} I)x, y) = ((\bar{P}(A)-\bar{\zeta} I)x, y).$$

Since $D(A^n)$ is dense in $\Pi_\kappa$ by Lemma 3.4, we have $(\bar{P}(A)-\bar{\zeta} I)x=\theta$, the zero vector. As $\bar{\zeta}$ is not a proper value of $\bar{P}(A)$, $x$ must be the zero vector $\theta$. Therefore $(P(A)-\zeta I)D(A^n)$ is dense in $\Pi_\kappa$.

We shall show $(P(A)-\zeta I)D(A^n)=\Pi_\kappa$. Let us define an operator $U$ in the $\Pi_\kappa$ by the formulae:

$$y = (P(A)-\zeta I)x, \qquad Uy = (\bar{P}(A)-\bar{\zeta} I)x$$

for all $x \in D(A^n)$. Clearly $U$ is an isometric operator with dense domain in $\Pi_\kappa$. The operator $U$ is bicontinuous by Theorem 4.3 in [1, §15]. Since the operators $(A-\zeta_i)^{-1}$, $i=1, 2, \ldots, n$ are continuous by Theorem 3.1, the operator $(P(A)-\zeta I)^{-1}=\prod_{i=1}^n (A-\zeta_i)^{-1}$ is also continuous. As $\bar{P}(A)=P(A)^*$ is a closed operator, it follows that $U$ and $P(A)-\zeta I$ are also closed operators. Applying Theorem 2.3 we conclude that $(P(A)-\zeta I)D(A^n)=D(U)$ is a subspace. Since it is dense in $\Pi_\kappa$, it can only be the whole space $\Pi_\kappa$. As $P(A)-\zeta I$ is a closed operator, the operator $P(A)$ must be a closed operator. It thus follows from Lemma 3.6 that $P(A)=P(A)^{**}$. Hence by similar arguments we have $(\bar{P}(A)-\bar{\zeta} I)D(A^n)=\Pi_\kappa$.

We shall show that $(A-\zeta_1)D(A)=\Pi_\kappa=(A-\bar{\zeta}_1)D(A)$. It is sufficient to prove $\prod_{i=2}^n (A-\zeta_i)D(A^n)\supset D(A)$. Since $\prod_{i=1}^n (A-\zeta_i)D(A^n)=\Pi_\kappa$, for any $x \in D(A)$ there exists $x' \in \prod_{i=2}^n (A-\zeta_i)D(A^n)$ such that $(A-\zeta_1)x'=(A-\zeta_1)x$, that is $(A-\zeta_1)(x'-x)=\theta$. Since $\zeta_1$ is not a proper value of $A$, we must have $x=x'$.

We shall show that $A=A^*$. Since $D(A)\supset D(A^n)$ and $\mathrm{cl}\,(D(A^n))=\Pi_\kappa$, $A^*$ exists. It is obvious $A^*\supset A$; therefore it is sufficient to prove $D(A^*)\subset D(A)$. Since

$(A-\zeta_1)D(A)=\Pi_\kappa$, for any $x\in D(A^*)$ there exists $y\in D(A)$ such that $(A^*-\zeta_1)x=(A-\zeta_1)y$, that is $(A^*-\zeta_1)(x-y)=\theta$. It follows that $((A^*-\zeta_1)(x-y),z)=0$ for all $z\in D(A)$. Hence $((x-y),(A-\bar{\zeta}_1)z)=0$ for all $z\in D(A)$. As $(A-\bar{\zeta})D(A)=\Pi_\kappa$, we have $x-y=\theta$, that is $x=y$. So we have $D(A^*)=D(A)$ and $A=A^*$. The theorem is completely proven.

THEOREM 3.8. *Let $A$ be a symmetric operator in $\Pi_\kappa$ and let $P(\lambda)$ be a real polynomial of degree greater than one. Then $P(A)$ is a maximal symmetric operator if and only if $A$ is self-adjoint.*

**Proof.** If $A$ is self-adjoint, the operator $P(A)$ must be self-adjoint, that is maximal, by Theorem 3.7. Now let $P(A)$ be maximal and let $\tilde{A}$ be a maximal symmetric extension of the operator $A$. Then $P(A)=P(\tilde{A})$ must hold. If $\tilde{A}$ is self-adjoint, then $P(A)$ is self-adjoint and consequently $A$ is self-adjoint by Theorem 3.7. If $\tilde{A}$ is not self-adjoint, we shall show $P(A)=P(\tilde{A})$ can not be maximal. If $P(\tilde{A})$ were maximal there exists a nonreal number $\zeta$ which is not a proper value of $P(\tilde{A})$ such that

$$(P(\tilde{A})-\zeta)D(\tilde{A}^n) = \Pi_\kappa = \prod_{i=1}^{n}(\tilde{A}-\zeta_i)D(\tilde{A}^n). \tag{3.8}$$

It follows that the roots of $P(\lambda)-\zeta$ are not proper values of $A$. Since $P(\lambda)$ is a real polynomial of degree at least two and since $\zeta$ is a nonreal number, there exists at least one root of the polynomial $P(\lambda)-\zeta$ in both the upper and the lower half of the complex plane. It thus follows that there exists a root $\zeta_{i0}$ such that $(\tilde{A}-\zeta_{i0})D(A)\neq\Pi_\kappa$. Hence we have

$$\prod_{i=1}^{n}(\tilde{A}-\zeta_i)D(\tilde{A}^n)\subset(\tilde{A}-\zeta_{i0})D(\tilde{A})\neq\Pi_\kappa.$$

This contradiction implies that $A$ must be a self-adjoint operator. The theorem is proven.

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